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# Stability of Two-Phase Flow Calculation Using Two-Fluid Models*

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Two-fluid modeling of two-phase flow may yield a system of partial differential equations having complex characteristics; this results in a mathematically ill-posed initial-value problem. Despite the fact that no finite-difference method for solving such a problem can be stable in the usual sense, finite-difference solution of two-fluid models is in widespread use. We investigate the numerical behavior of one such set of difference equations, derive conditions under which solutions appear to be well behaved, and offer a physical interpretation.

Considerable attention has recently been given to calculation methods for transient multifluid flow problems. A semi-implicit multifluid Eulerian finite-difference method was first proposed by Harlow and Amsden [1, 2], and has since been improved by Liles and Reed [3] with variations by Stewart [4] and Solbrig *et al.* [5] among others. Multifluid modeling and numerical calculation appear to have important applications. One is in reactor safety analysis, where effort is being made to better understand and predict two-phase flow behavior through the potentially more rational approach of multifluid modeling [6, 7]. By two-fluid equations we shall mean a multifluid model for two-phase flow consisting of mass, momentum, and energy conservation equations for each of the phases under a common pressure field, with nondifferential exchange terms to describe the exchanges of mass, momentum, and energy between phases.

As is by now well known [8–10, 19], this system of six partial differential equations (for one space dimension) has the mathematical property that its six characteristic roots are not all real, as would be the case for a hyperbolic system. Instead, two of the roots are complex. This implies that solutions of an initial-value problem for the equations do not depend continuously on initial values, so the initial-value problem is not well posed [11, p. 80; 12]. Some researchers have concluded that the differential equations should be modified to render all characteristic roots real [8, 10]. Nevertheless, numerous approximate finite-difference calculations are being carried out with the several numerical methods in existence for two-fluid equations without

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observable symptoms of ill-posedness. It is not our intent in this paper to decide what approach is best. Rather, we wish to understand why, and more precisely under what conditions, a particular popular set of two-fluid finite-difference equations can be expected to give reasonable numerical results in spite of the ill-posedness of the differential equations.

For simplicity, we shall limit ourselves in this paper to two-fluid equations in one space dimension for mass and momentum conservation only; these four equations contain the essential features for our purposes. The first section of this paper will review known facts, including the characteristic roots of the two-fluid equations, and an elementary stability analysis for a single transport equation with complex characteristic velocity. We indicate our intent to avoid using the word stable in the usual mathematical sense, but to consider a special definition of well-behaved solutions based on physical intuition. In the second section we examine the numerical behavior of two-fluid finite-difference equations without interphase exchange terms, finding instability due to complex characteristics. However, the shortest wavelengths represented in the finite-difference solution, which are low multiples of the mesh spacing $\Delta z$, usually *are* well behaved; this fact is significant because in a complicated, realistic transient calculation longer-wavelength instabilities may be difficult to detect. In the following section we show how some longer wavelengths may also be well behaved if the two-fluid model contains a large enough momentum exchange, and if $\Delta z$ is not too small. An approximate criterion relating momentum transfer and mesh size is presented, which admits a meaningful physical interpretation. We then discuss briefly the results of some numerical calculations; the final section summarizes our results.

## Background

We begin with the four partial differential equations of mass and momentum conservation of the two-fluid model. These equations might describe isothermal two component two-phase flow in a straight pipe, for example.

$$\frac{\partial \alpha \rho_v}{\partial t} + \nabla \cdot \alpha \rho_v u_v = 0, \tag{1}$$

$$\frac{\partial (1-\alpha)\,\rho_l}{\partial t} + \nabla \cdot (1-\alpha)\,\rho_l u_l = 0, \tag{2}$$

$$\alpha \rho_v \left[\frac{\partial u_v}{\partial t} + u_v \nabla \cdot u_v\right] + \alpha \nabla P = K(u_l - u_v), \tag{3}$$

$$(1-\alpha)\,\rho_l \left[\frac{\partial u_l}{\partial t} + u_l \nabla \cdot u_l\right] + (1-\alpha)\,\nabla P = K(u_v - u_l). \tag{4}$$

The right sides of the momentum equations are equal and opposite interphase momentum transfer terms in a general form used by several authors [1–5].

To find the four characteristic roots of these equations in one space dimension we form the characteristic determinant of the system [13, p. 170ff.]. We shall assume throughout for simplicity that the liquid is incompressible compared to the vapor (reasonable unless $\alpha$ is very small), so the characteristic roots $\mu$ satisfy

$$\alpha\rho_l(\mu - u_l)^2 + (1 - \alpha)\,\rho_v(\mu - u_v)^2 - \alpha\rho_l c_v^{-2}(\mu - u_v)^2(\mu - u_l)^2 = 0, \qquad (5)$$

where $c_v^{-2} = \partial\rho_v/\partial P$. Multiplying by $c_v^2$ and reorganizing, we find there are two real roots, one slightly less than $u_v - c_v$ and one slightly greater than $u_v + c_v$. From the slope of this polynomial, any further real roots would have to lie between these two, but examination of the polynomial itself shows this to be impossible provided $u_v \neq u_l$. Hence there must be a pair of complex conjugate roots. If $u_v, u_l \ll c_v$, these complex roots are approximately

$$\mu \simeq \frac{u_l + \epsilon u_v}{1 + \epsilon} \pm i\frac{\epsilon}{1 + \epsilon}(u_v - u_l),$$

where $\epsilon^2 = (1 - \alpha)\,\rho_v/\alpha\rho_l$.

The effect of complex characteristic velocities on finite-difference equations can be seen by reviewing a simple example:

$$\frac{\partial\phi}{\partial t} + u(1 + \epsilon i)\frac{\partial\phi}{\partial z} + K\phi = 0. \qquad (6)$$

We assume that $u$, $\epsilon$, $K$ are nonnegative constants. Let this equation be approximated by the difference equation

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + u(1 + \epsilon i)\frac{\phi_j^n - \phi_{j-1}^n}{\Delta z} + K\phi_j^{n+1} = 0, \qquad (7)$$

where subscript $j$ refers to spatial node and $n$ indicates time step. This difference equation uses donor-cell differencing for the explicit convective term and implicit treatment of the damping term $K\phi$. Applying the von Neumann linear stability analysis method [14–16], we consider a Fourier component of $\phi$, namely, $\exp(ikj\,\Delta z)$, where $k$ is a wave number. Note that $k$ takes on the discrete values $\pi/(n\,\Delta z)$, $n = 1, 2, \ldots, N$, where $N$ is the total number of intervals in the spatial mesh grid. From one time step to the next, the magnitude of this component will be found to be multiplied by

$$\lambda = (1 + K\,\Delta t)^{-1}\left[1 - \frac{u\,\Delta t}{\Delta z}(1 + \epsilon i)(1 - \exp(-ik\,\Delta z))\right].$$

If $|\lambda| > 1$, then the corresponding Fourier component of $\phi$ can grow geometrically, while if $|\lambda| \leqslant 1$ for all $k$, the solution remains bounded for all time steps.

When $K = \epsilon = 0$, the locus of values of $\lambda$ for all $n$ is a sequence of points on a circle in the complex plane with radius $u\,\Delta t/\Delta z$ centered at $1 - u\,\Delta t/\Delta z$. The circle is tangent to the vertical at the point 1; if $u\,\Delta t/\Delta z < 1$ it lies inside the unit circle. Figure 1

illustrates the locus of $\lambda$, with points approaching the point 1 as $n$ becomes large. Since with $K = \epsilon = 0$ the solutions of (6) are just traveling waves, approximate solutions should not have geometrically growing components, and $|\lambda| \leqslant 1$ is required for stable numerical solution. This means $u\,\Delta t/\Delta z < 1$.

Now if $K = 0$ but $\epsilon \neq 0$, then the previous locus is effectively dilated by $(1 + \epsilon^2)^{1/2}$ and rotated by an angle arctan $\epsilon$, both about the point 1 in the complex plane. If $\epsilon$ were negative, the result might still be inside the unit circle, but for any $\epsilon > 0$, no matter how small, there is an $n$ large enough that the corresponding $\lambda$ will be tilted outside the unit circle (even if $u\,\Delta t/\Delta z < 1$). For such $n$, the corresponding Fourier mode

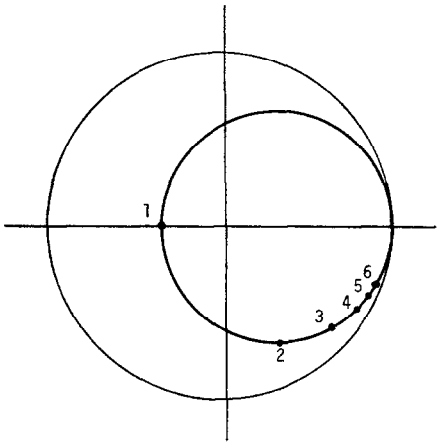


FIG. 1. Locus of $\lambda$ for $n = 1, 2, 3, \ldots$.

will grow geometrically; the original differential equation also has such geometrically growing Fourier modes. Note, however, that the numerical solution may not have growing modes if there are only a few nodes.

Finally if $K > 0$, then the above locus is further changed by a contraction toward the origin $(1 + K\,\Delta t)^{-1}$. Evidently, if $K$ is large enough, the entire locus may lie inside the unit circle even for large $n$.

This simple example, known to many researchers (Wendroff [17]), shows features we shall find for two-fluid equations. Donor-cell differencing of the convective term with $u\,\Delta t/\Delta z < 1$ has a stabilizing effect on the highest-frequency modes even when the convecting velocity is complex. The destabilizing effect of the complex velocity on low-frequency modes may be compensated, for a given mesh size, by a large enough damping term. As the mesh is refined, $n \to \infty$, $\Delta z \to 0$, $\Delta t \to 0$, and $1 + K\,\Delta t \to 1$, so the finite-difference method must eventually give geometrically growing modes. With $u\,\Delta t/\Delta z < 1$, the first manifestation of this will be at wavelengths many times $\Delta z$.

The following analysis for the two-fluid equations leads to similar conclusions. The role of the damping term will be played by the momentum transfer between phases. An important result will be a physically based condition for well-behaved solutions

with the finite-difference method; this condition relates the momentum exchange term to the mesh size.

In this paper we shall not refer to the term stability in the usual mathematical sense, which requires certain nice behavior uniformly as $\Delta z$, $\Delta t \to 0$. The two-fluid equations with complex characteristics cannot be stable in that sense. Referring back to the physical flow problem we are attempting to model, we expect that the two-fluid equations should be capable of describing steady flows; this would be impossible with geometrically growing modes. Thus our ad hoc definition of *well-behaved* solutions will be that for a given mesh; local linear stability analysis should give $|\lambda| \leqslant 1$ for all modes.

## Two-Fluid Equations without Exchange Terms

We shall analyze the behavior of certain finite-difference approximations to (1)–(4). These difference equations are partially implicit, in such a way that sonic propagation effects are treated implicitly, while convection phenomena are handled explicitly, so that the time step $\Delta t$ is limited by $\Delta t \leqslant \min(\Delta z/u_v, \Delta z/u_l)$. This type of difference equation has been used by several authors [1–3]. A convenient spatial mesh has pressures and void fractions defined at integral modes $j$, while velocities $u_v$, $u_l$ are defined at half-nodes $j + \frac{1}{2}$. We refer to old time values by superscript $n$, new time values by $n + 1$. The difference equations are

$$(\alpha_j^{n+1}\rho_{v\,j}^{n+1} - \alpha_j{}^n\rho_{v,j}^n)/\Delta t + (\alpha_{j-1}^n\rho_{v,j-1}^n u_{v,j-1/2}^{n+1} - \alpha_j{}^n\rho_{v,j}^n u_{v,j+1/2}^{n+1})/\Delta z = 0, \tag{8}$$

$$((1 - \alpha_j^{n+1})\,\rho_{l,j}^{n+1} - (1 - \alpha_j{}^n)\,\rho_{l,j}^n)/\Delta t + ((1 - \alpha_{j-1}^n)\,\rho_{l,j-1}^n u_{l,j-1/2}^{n+1} - (1 - \alpha_j{}^n)\,\rho_{l,j}^n u_{l,j+1/2}^{n+1})/\Delta z = 0, \tag{9}$$

$$\alpha_j{}^n\rho_{v\,j}^n[(u_{v,j+1/2}^{n+1} - u_{v,j+1/2}^n)/\Delta t + u_{v,j+1/2}^n(u_{v,j+1/2}^n - u_{v,j-1/2}^n)/\Delta z] + \alpha_j{}^n(P_{j+1}^{n+1} - P_j^{n+1})/\Delta z = K_{j+1/2}(u_{l,j+1/2}^{n+1} - u_{v,j+1/2}^{n+1}), \tag{10}$$

$$(1 - \alpha_j{}^n)\,\rho_{l,j}^n[(u_{l,j+1/2}^{n+1} - u_{l,j+1/2}^n)/\Delta t + u_{l,j+1/2}^n(u_{l,j+1/2}^n - u_{l,j-1/2}^n)/\Delta z] + (1 - \alpha_j{}^n)(P_{j+1}^{n+1} - P_j^{n+1})/\Delta z = K_{j+1/2}(u_{v,j+1/2}^{n+1} - u_{l,j+1/2}^{n+1}). \tag{11}$$

The divergence term in the mass equations and the convective term in the momentum equations involve donor-cell differencing, and the above equations are for $u_v$, $u_l > 0$. In this section we shall consider the case $K = 0$.

To apply the von Neumann method of local linear stability analysis, we expand the differences above in terms of differences of four basic variables, say, $\rho_v$, $\alpha$, $u_v$, $u_l$. We then treat the coefficient of those individual differences as constant (hence "local

linear"). Considering a Fourier mode leads to the amplification matrix (cf. [16, p. 70])

$$\begin{bmatrix} \alpha(\lambda - 1 + \tilde{u}_v) & \rho_v(\lambda - 1 + \tilde{u}_v) & \lambda \dfrac{\Delta t}{\Delta z} \alpha \rho_v i k' & 0 \\ 0 & -\rho_l(\lambda - 1 + \tilde{u}_l) & 0 & \lambda \dfrac{\Delta t}{\Delta z}(1 - \alpha)\rho_l i k' \\ \lambda \dfrac{\Delta t}{\Delta z} c_v^2 i k' & 0 & \rho_v(\lambda - 1 + \tilde{u}_v) & 0 \\ \lambda \dfrac{\Delta t}{\Delta z} c_v^2 i k' & 0 & 0 & \rho_l(\lambda - 1 + \tilde{u}_l) \end{bmatrix}, \tag{12}$$

where the columns from left to right correspond to the variables $\rho_v$, $\alpha$, $u_v$, $u_l$; $k$ is again the wavenumber, $k = \pi/n\,\Delta z$, and we have abbreviated $k' = 2\sin \frac{1}{2}k\,\Delta z$, $\tilde{u}_v = (u_v\,\Delta t/\Delta z)[1 - \exp(-ik\,\Delta z)]$, and $\tilde{u}_l = (u_l/u_v)\,\tilde{u}_v$. The eigenvalues $\lambda$ of (12) satisfy

$$\begin{aligned} \lambda^2 C_n^2[(\lambda - 1 + \tilde{u}_l)^2 + (1 - \alpha)\,\rho_v/(\alpha\rho_l)(\lambda - 1 + \tilde{u}_v)^2] \\ + (\lambda - 1 + \tilde{u}_v)^2(\lambda - 1 + \tilde{u}_l)^2 = 0, \end{aligned} \tag{13}$$

where $C_n = (c_v\,\Delta t/\Delta z)(2\sin(\pi/2n))$.

First consider the high-frequency behavior. We note that the scheme (8)–(11) is particularly advantageous when $\Delta t$ is nearly equal to the convective limit value $\min(\Delta z/u_v, \Delta z/u_l)$, but much larger than $\Delta z/c_v$, in other words, $u_v, u_l \ll c_v$. This implies $c_v\,\Delta t/\Delta z \gg 1$, and for small $n$, $C_n \gg 1$. Then (13) has two roots of magnitude roughly $C_n^{-1} \ll 1$, and another two which approximately satisfy

$$\lambda - 1 + \tilde{u}_l \simeq \pm i\epsilon(\lambda - 1 + \tilde{u}_v),$$

where

$$\epsilon^2 = (1 - \alpha)\,\rho_v/\alpha\rho_l\,;$$

hence

$$\lambda \simeq 1 - \tilde{u}_l(1 \pm i\epsilon u_v/u_l)(1 \pm i\epsilon)^{-1}.$$

This is the usual locus of points on a circle of radius $u_l\,\Delta t/\Delta z$ touching the point 1 in the complex plane, but tilted by an angle $\pm$ arctan $(\epsilon u_v/u_l)$ about the point 1, then back through an angle $\mp$ arctan $\epsilon$. The net dilation is bounded by $1 + |u_v/u_l|$. Clearly points on this circle with *small n* (i.e., away from the point 1) need not be tilted outside the unit circle if $u_l\,\Delta t/\Delta z < 1$ and $u_v\,\Delta t/\Delta z < 1$. We conclude that if the usual convective limit on the time step is obeyed, the highest-frequency modes (i.e., wavelengths small multiples of $\Delta z$) should not grow geometrically, even for the ill-posed two-fluid problem without momentum exchange.

Now turning to the low-frequency modes, consider (13) as $n$ becomes large. Since

$C_n \to 0$ in the limit as $n \to \infty$, we see that the roots of (13) tend toward $\lambda = 1 - \tilde{u}_v$ (twice) and $\lambda = 1 - \tilde{u}_l$ (twice). To be more precise, let $\lambda = 1 - \tilde{u} + \delta$; we can approximate $\delta$ for large $n$ by rewriting (13) in terms of $\delta$ and keeping only terms up to second order in $\delta$. The resulting quadratic polynomial in $\delta$ has roots

$$\frac{\delta}{[(1 - \tilde{u}_l)(\tilde{u}_v - \tilde{u}_l)]} \simeq \frac{\epsilon^2 \pm i\epsilon(1 - (u_v - u_l)^2/c_v^2)^{1/2}}{1 + \epsilon^2 - (u_v - u_l)^2/c_v^2}.$$

Again using (8)–(11) it is appropriate to assume that $| u_v - u_l | \ll c_v$, so

$$\lambda \approx (1 - \tilde{u}_l)\left[1 - i(\tilde{u}_v - \tilde{u}_l)\frac{\pi \Delta t}{n \Delta z}\epsilon\left(\frac{\epsilon \pm i}{1 + \epsilon^2}\right)\right].$$

Since $| 1 - \tilde{u}_l | = 1 - O(n^{-2})$ and is multiplied by a quantity having magnitude $1 + O(n^{-1})$, where $O(n^{-1})$ is positive for one choice of the $\pm$ above, one root $\lambda$ must lie outside the unit circle for large $n$. A similar result follows from a perturbation $\lambda = 1 - \tilde{u}_v + \delta$. Hence for (8)–(11) with $K = 0$, wavelengths with large $n$ will be unstable.

We remark that there are two interpretations of augmenting $N$, the total number of mesh intervals, for this problem. One is adding physical length to the problem and holding $\Delta z$ fixed. However, since the equations with $K = 0$ are invariant under a uniform dilation of both space and time, another interpretation of augmenting $N$ is keeping the total physical length fixed and shrinking $\Delta t$ and $\Delta z$, i.e., refining the mesh.

Our conclusions for the case $K = 0$ are that numerical calculations with the two-fluid difference equations (8)–(11) and very few spatial nodes may have well-behaved solutions. As the mesh is refined, instabilities will appear, likely of wavelength many times $\Delta z$.

## Two-Fluid Equations with Momentum Exchange

Now consider the finite-difference equations (8)–(11) with $K > 0$. Note that in (10) and (11) the velocities in the momentum equation are treated implicitly, thus avoiding any upper limit on $\Delta t$ due to strong momentum coupling. The new amplification matrix differs from (12) only in the lower right corner $2 \times 2$ which becomes

$$\begin{matrix} \rho_v(\lambda(1 + \kappa) - 1 + \tilde{u}_v) & -\lambda\rho_v\kappa \\ -\lambda\rho_v\kappa & \rho_l(\lambda - 1 + \tilde{u}_l) + \lambda\rho_v\kappa \end{matrix}$$

where $\kappa = K \Delta t/\rho_v$. (Implicit treatment of velocities in the momentum exchange is desirable when $\kappa > 1$.) The new characteristic polynomial thus obtained must again be analyzed to see if $| \lambda | \leqslant 1$.

First consider the high-frequency behavior and let $n$ be small. Assuming $\kappa \gg 1$ (i.e., that implicit treatment of momentum exchange is very desirable), and also as

before that $C_n \gg 1$, we find the characteristic polynomial has two roots of magnitude roughly $C_n^{-1}$, one root of magnitude roughly $\kappa^{-1}$, and one approximately equal to $1 - (\tilde{u}_l + \epsilon^2\tilde{u}_v)/(1 + \epsilon^2)$, where again $\epsilon^2 = (1 - \alpha)\,\rho_v/\alpha\rho_l$. This is again inside the unit circle if $\Delta t$ does not exceed the convective limit. Note also that all four roots are inside the unit circle for small $n$ and very small $\kappa$, because we have skown this for $\kappa = 0$ and the roots depend continuously on $\kappa$. Although we have only argued for large $\kappa$ (the case most appropriate to implicit treatment of momentum exchange) and small $\kappa$, it is reasonable to expect that wavelengths with small $n$ do not grow for any value of $\kappa$, since equations with an implicitly treated damping term should not have more tendency to growing modes than without.

The low-frequency behavior comes again from examining the limiting case $n \to \infty$, and then considering a perturbation of low order in $n^{-1}$. A remark about interpreting $N \to \infty$ is in order here, as the nondifferential term $K(u_v - u_l)$ makes a difference. If $N$ means to refine the mesh, then ultimately $\kappa = K\,\Delta t/\rho_v$ must go to zero. The term $K(u_v - u_l)$ has a built-in physical length. We will start, however, by letting $n \to \infty$ but retaining $\kappa$. This could be interpreted as refining the mesh and supposing that at some point all higher-order terms in $(\Delta z)^{-1}$, $(\Delta t)^{-1}$ are negligible except $\kappa$; or it could be interpreted as holding $\Delta z$ fixed and adding physical length to the problem considered.

The limiting equation for fixed $\kappa$ and $n \to \infty$ is

$$(\lambda - 1 + \tilde{u}_v)(\lambda - 1 + \tilde{u}_l)[(\kappa\lambda + \lambda - 1)(\rho_l(\lambda - 1) + \lambda\rho_v\kappa) + \lambda^2\rho_v\kappa^2] = 0.$$

The two roots of the quadratic polynomial in brackets can be shown to be real, positive, and strictly less than one. Hence for all $n$ sufficiently large, the corresponding roots must lie within the unit circle.

For the remaining two roots let $\lambda = 1 - \tilde{u}_v + \delta$ as before, substitute into the original polynomial equation, and keep only first order terms in $\delta$. Furthermore, neglect higher-order terms in $n^{-1}$. Finally, suppose that

$$\rho_l \mid \tilde{u}_l - \tilde{u}_v \mid \ll 2\rho_v\kappa. \tag{14}$$

With these assumptions the one remaining root is approximately

$$\lambda = (1 - \tilde{u}_v)\left(\frac{\kappa + C_n^{\,2}}{\kappa + 2C_n^{\,2}}\right)$$

which is inside the unit circle. The same result follows with $\lambda = 1 - \tilde{u}_l + \delta$, implying that long-wavelength modes are well behaved.

Condition (14) is the key to this long-wavelength behavior. It implies that momentum exchange must be larger than a certain minimum value in order to avoid geometrically growing long-wavelength modes. We will now show that this minimum momentum transfer relates to the mesh size in a physically meaningful way.

First, note that (14) for large $n$ is equivalent to

$$K(n\,\Delta z)/\rho_v \gg \mid u_v - u_l \mid (\rho_l/\rho_v). \tag{15}$$

As remarked above, $K$ contains a built-in physical length. For example, one form proposed for $K$, on the basis of analogy with flow of an isolated nondeforming bubble in liquid or droplet in vapor, is essentially ([1, 4])

$$K \simeq C_D \rho_v \mid u_v - u_l \mid / r,$$

where $r$ is the droplet or bubble radius and $C_D$ is a drag coefficient on the order of one. We then find that (14) reduces to

$$n \Delta z \gg (\rho_l / \rho_v C_D) r. \tag{16}$$

In other words, the wavelength $n \Delta z$ will not have a growing mode if it is larger than a certain multiple of the radius of an individual bubble or droplet.

To review, we have shown that a two-fluid calculation using difference equations (8)–(11) will have nongrowing high-frequency components, while low-frequency components with large $n$ and satisfying (15) for a given mesh will also be well behaved.

These arguments do not guarantee there is no growing mode. One might think it plausible to require (15) down to $n = 1$ to cover the gap between low and high frequencies, but of course (15) was used specifically for large $n$. We have shown that, provided $K \Delta z$ is large enough, momentum transfer stabilizes long wavelengths which could otherwise have geometrically growing modes.

## Numerical Examples

Results of many calculations with schemes similar to (8)–(11) have shown that in fact they appear to be well behaved at all wavelengths (that is no geometrically growing modes) in a large number of practical cases. Furthermore, if $K$ is reduced, growing oscillations of long wavelength may appear.

We have carried out some trial calculations to illustrate this, choosing as an example the flow of a steam–water mixture in a horizontal 20-mm-diameter tube, including wall friction and a momentum exchange law proposed for annular flow (Wallis [18, p. 321]). Although we used the equations of state of liquid water and steam, we assumed no phase change. Boundary conditions were $\alpha = 0.5$, $P = 150.1$ bars, and saturation temperatures at the inlet, and $P = 150$ bars at the outlet. The transient calculation began from initial conditions of constant $\alpha$, $T_l$, $T_v$ equal to the inlet values, and $u_v = u_l$ and $G = \alpha \rho_v u_v + (1 - \alpha) \rho_l u_l = 10^3$ kg/m$^2$/sec. Calculations were performed for a tube 15 mm long with 5, 10, 15, 25, 50, and 75 equal steps between boundary pressure points. The time step was 0.12 msec with five space steps, and proportional to $\Delta z$ for the other cases. Table I shows the values of $G$ at the inlet after 6 msec. The final column gives the slope between successive data points in a plot of $G(t = 6$ msec) versus $\Delta z$.

We observe that for a first-order accurate approximation to a well-posed problem, we would expect the figures in the slope column to tend toward a constant value as the

TABLE I

A sample two-fluid calculation

| Number of nodes | Mass flow rate $G$ after 6 msec | Slope of $G$ versus $\Delta z$ (relative units) |
|---|---|---|
| 5 | 3338.598171 | |
| | | 1.000 |
| 10 | 3338.891851 | |
| | | 0.984 |
| 15 | 3338.988149 | |
| | | 0.978 |
| 25 | 3339.064757 | |
| | | 0.975 |
| 50 | 3339.122022 | |
| | | 1.088 |
| 75 | 3339.143324 | |

number of nodes increases. In fact, the slopes decrease slightly in the first four instances, which would imply slightly less than first-order convergences. With 75 nodes, this trend is clearly broken. In fact, this calculation showed the first signs of a geometrically growing oscillation.

By substituting the value used for $K$ into (15) above and taking $\Delta z = 15$ mm/75 = 0.2 mm, we would find equality in (15) with $n \simeq 150$. In other words, using (15) with $n = 1$ as a criterion for $\Delta z$ would in this case be conservative by a factor of 150. We also note that in this case the smallest well-behaved mesh spacing is more than 50 times smaller than the tube diameter.

## Conclusions

We have shown that numerical calculation with a two-fluid finite-difference model of two-phase flow can be well-behaved provided there is sufficient momentum transfer between phases, and the spatial mesh is not too fine. This is in spite of the fact that the two-fluid differential equations have complex characteristic roots, so that the continuous initial-value problem is ill-posed, and calculation with a sufficiently fine mesh must be unstable. Moreover, we have given a mesh size criterion with a physical interpretation: for momentum exchange models based on bubble or droplet flow, the critical mesh size is a multiple of the bubble or droplet radius. In other words, approximate calculations can be well behaved provided one does not attempt to resolve phenomena finer than the scale implied in the momentum transfer law. This is physically reasonable, since the two-fluid model in question is intended to approximate bubble or droplet regime flow, for example, by ignoring fluctuations caused by an individual bubble or droplet. Thus for coarse modeling of two-phase flow, the model is mathematically ill-posed, but approximate calculation is well behaved (in our ad hoc sense) if one stays on the appropriate scale.

On the other hand, in the case of no momentum transfer (e.g., modeling separated

flow), variation in void fraction refers to a shifting liquid–vapor contact surface. Any approximate solution allowing void fraction to vary is of necessity a fine-scale resolution when $K = 0$. In this case, a well-posed problem is obtained by including the effect of surface tension (Ramshaw and Trapp [10]). The same might, in principle, be done for other flow regimes if very fine resolution were demanded, and all liquid–vapor interfaces could be followed in detail. Thus for fine modeling, the appropriate model is different, mathematically well-posed as an initial value problem, and describes the physical Helmholtz instability.

The conclusions seem to suggest using the two-fluid model (1)–(4) when coarse modeling is appropriate, in spite of ill-posedness. However, there are reservations. One has to do with the nature of the instability when the mesh becomes too fine. With the numerical scheme considered, instability associated with complex characteristics will not be the $2\,\Delta z$ wavelength growing oscillation of purely numerical instability. In a complex, realistic transient calculation, a longer-wavelength instability may have significant effects without its presence being obvious.

Furthermore, our analysis merely suggests reasonable physical limits on the applicability of Eqs. (1)–(4). Whether they, in fact, model reality is a question for confrontation with experiments.

It would be interesting to have a numerical method which exhibits a readily apparent high-frequency instability when $K\,\Delta z$ becomes too small. In fact, the method proposed by Stewart [4] was found by numerical experiment to have this property, although the reason for this could not be clearly discerned by analysis.

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